Analysis of Bifurcation and Chaos by Describing Function Method

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Abstract In this paper it is shown that the bifurcation and chaotic behavior in a nonlinear system can be analyzed by using the describing function method. The proposed method to predict bifurcation or chaos can be applied for a nonlinear system of any order and also applied for a nonlinear system with time lag.

1. Introduction

In this paper the author's works to predict bifurcation and chaos using the describing function method published hitherto in several journals and proceedings are rewritten and summarized. The describing function method is well known in the control theory [1] and the application of this method to the bifurcation and chaotic phenomena have also been studied [2],[3]. The methods studied in this paper will be much easier to apply by comparing with other methods. Since the analysis is made in the frequency domain, it is possible to analyze a nonlinear system of any order or system with time lag.

Throughout this paper a nonlinear control system shown in Fig.1 will be considered, where G(s) is a transfer function of a linear part and it is assumed that G(s) has no poles in the RHP and has the property of a low-pass filter, and f(y) is a nonlinear function.

Now let

$$G(s) = N(s) / D(s), \tag{1}$$

where $D(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ and $N(s) = 1$

This system can also be represented by a differential equation form as

$$D(p)y + f(y) = r(t), \qquad p = d/dt \tag{2}$$

or

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3, \cdots, \dot{x}_{n-1} = x_n,$$
$$\dot{x}_n = -(a_1 x_n + \cdots + a_n x_1 + f(y)) + r(t) \tag{3}$$
$$y = x_1$$

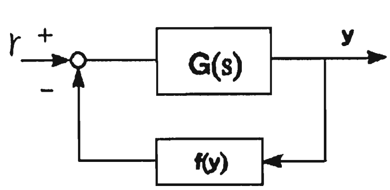


Fig.1 Control system

The purpose of this paper is to show how to obtain the parameter regions of bifurcation and chaos by using the describing function method.

2. Analysis of bifurcation

It is well known that the output amplitude to a sinusoidal input in a nonlinear system jumps to higher or lower magnitude by a slight change of the input frequency if some conditions are satisfied. This phenomenon is called jump resonance in control or mechanical engineering. This phenomenon can be considered as the bifurcation phenomenon. The following investigation is to analyze the jump resonance phenomenon as one of the bifurcation phenomena and it will be shown that the jump resonance occurs not only by a slight change of the input frequency but also by a slight change of the other parameters.

Let consider the periodic motion in this system. According to the assumptions, when the sinusoidal input $r(t) = R\sin\omega t$ is applied, the output of the system can be assumed to be sinusoidal, that is, it can be assumed as $y(t) = x_1(t) = X\sin(\omega t + \varphi)$. Let the describing function of $f(y)$ be $G_d(X)$. The closed loop transfer function can be represented by

$$Y(s) = R(s)/(G^{-1}(s) + G_d(X)) \tag{4}$$

where $Y(s)$ and $R(s)$ is the Laplace transformation of $y(t)$ and $r(t)$ respectively. Putting $s = j\omega$ and substituting

$$G^{-1}(j\omega) = \alpha(j\omega) + j\beta(j\omega), \quad \text{and} \quad G_d(X) = \xi(X) + j\eta(X) \tag{5}$$

into (4), we have

$$(R/X)^2 = (\alpha(\omega) + \xi(X))^2 + (\beta(\omega) + \eta(X))^2. \tag{6}$$

We put

$$F(X) = X^2\left(\ (\alpha + \xi)^2 + (\beta + \eta)^2\ \right) - R^2 = 0 \tag{7}$$

The function $F(X)$ will be an expression of the equilibrium state with respect to state variable X and parameters $R, \alpha, \beta, \omega, \cdots$ etc. So the saddle-node type bifurcation with respect to X occurs if $\partial F/\partial X = 0$. Eliminating X from (7) and equation of $\partial F/\partial X = 0$, the parameter region for bifurcation in the parameter space can be obtained. Practically it will be useful to show the region for bifurcation on the parameter plane by choosing two of parameters and keeping other parameters constant. For example, the $\alpha - \beta$ plane (i.e. $G^{-1}(j\omega)$ plane), the $R - \omega$ plane, and the $R - D$ plane, where D is a parameter of the nonlinear characteristics, can be considered. In Table 1, examples of parameter regions for bifurcation for various types of nonlinear characteristics are shown [4]. The solid (broken) line means that bifurcation occurs from a higher (lower) level to a lower (higher) level of the amplitude X [5].

Example 1

$$G(s)=\frac{1}{s^2+2\varsigma\omega_n s+\omega_n{}^2}$$

$$f(y)=y \quad |y|\le D, \qquad =+D \quad y>D, \quad =-D \quad y<-D,$$

The parameter region for bifurcation on the $R-D$ plane for $\varsigma=0.2$, $\omega_n=0.5$, and $\omega=1$ is shown in the right column in Table 1. The relation between X and D can be shown in Fig.2.

In the above analysis $N(s)$ is assumed to be unity in (1) for simplicity, however it will be clear that if $N(s)$ is a polynomial of s, the above method can be applied without any difficulty. In addition, if there exists a time lag element in $N(s)$ or $D(s)$, the above method can be applied.

Example 2

$$G(s)=\frac{1}{s^2+as+b+c\exp(-\tau s)}$$

$f(y)$ is the same as example 1.

If a=0.2, b=0.5, c=0.1, R=1.8, D=1 and τ=25, the bifurcation diagram can be shown in Fig.3 [6].

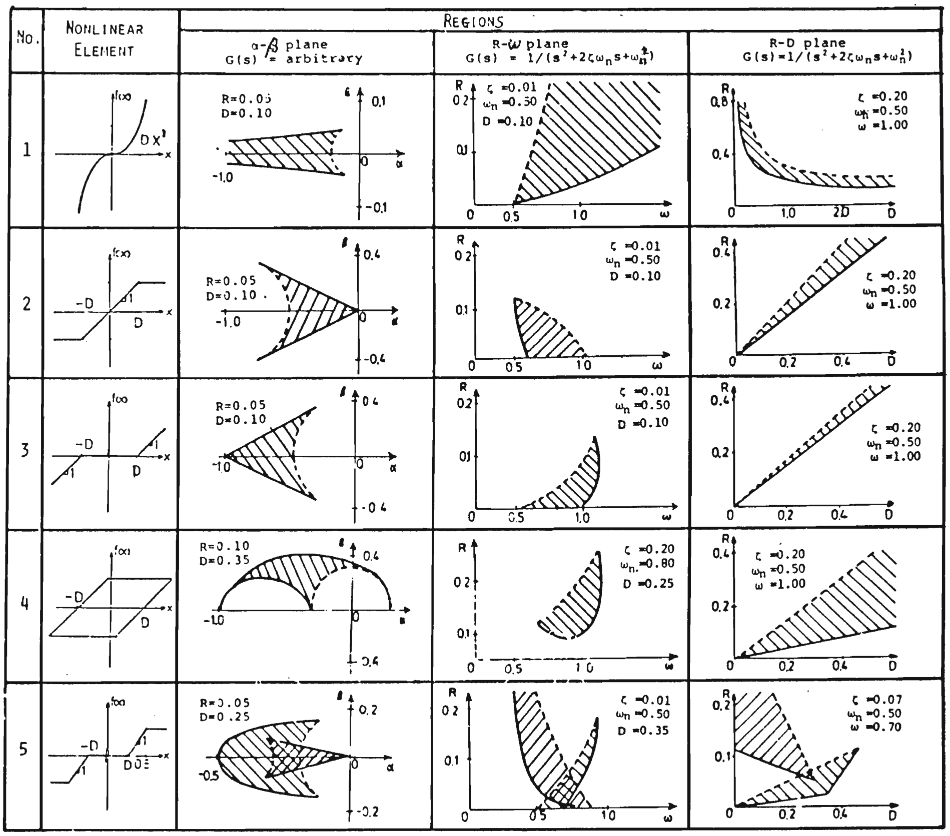
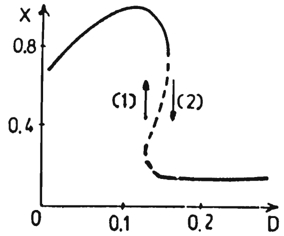


Table 1 Parameter regions for bifurcation for various nonlinear elements



Fig,2 Bifurcation in R-D plane

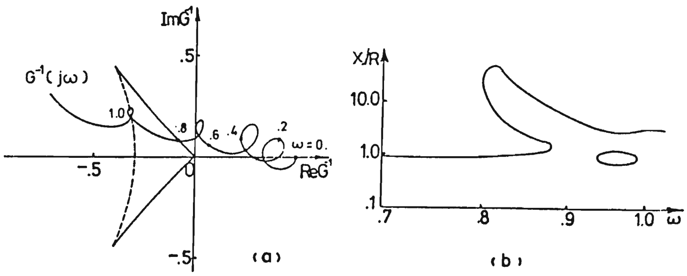


Fig.3 a) Bifurcation region b) bifurcation diagram

3. Analysis of Chaos

The following conjecture for the occurrence of chaos will be proposed, that is,

1) if there exist multiple equilibrium points and they are all unstable, and
2) if there exists an unstable limit surface, from where every trajectory goes away, around these equilibrium points,

then a trajectory starting from the inside of the limit surface does not converge to any equilibrium points and also does not diverge, and the trajectory may wander in the state space. It may be chaos.

This conjecture can be formulated by using the describing function method as following [7]. Instead of the existence of an unstable limit surface, firstly the existence of an unstable limit cycle will be studied.

Let a nonlinear function in Fig.1 be $f(x)=x^3-kx,\quad k>0$. From (3), the equilibrium points are given by

$$\begin{aligned} &P^0:\ x_1=x_2=\cdots=x_n=0 \\ &P^{\pm}:\ x_1=\pm\sqrt{k-a_n},\ x_2=x_3=\cdots=x_n=0 \end{aligned} \tag{8}$$

It can be shown that if $k\le a_n$, there exists only one stable equilibrium point P^0, and if $k>a_n$, it becomes unstable and two stable equilibrium points $P^{\pm}$ appear, and then, if k increases further, $P^{\pm}$ becomes unstable and two stable limit cycles around them appear. This condition is the case where three equilibrium points become all unstable.

Now let one of the stable limit cycle be represented by $x_1(t)=A+B_s\sin\omega t$. The bias component, A and the amplitude of a stable limit cycle, B_s and the amplitude of unstable limit cycle surrounding three equilibrium points, B_u can be calculated by using the describing function method as

$$
\begin{aligned}
A &= \pm\sqrt{\frac{1}{5}(k+g_0-2\ell_0)} \\
B_s &= \sqrt{\frac{4}{15}(2k-3g_0+\ell_0)} \\
B_u &= \sqrt{\frac{4}{3}(k-\ell_0)}
\end{aligned}
\tag{9}
$$

where

$$g_0 = 1/G(0),\quad \ell_0 = 1/\mathrm{Re}G(j\omega_p),\quad \mathrm{Im}G(j\omega_p)=0.$$

In Fig.4 the schematic location diagram of limit cycles is shown. In order for the unstable limit cycle to encircle two stable limit cycles,

$$B_s + |A| \le B_u \tag{10}$$

must be satisfied. Substituting (9) into (10), we have

$$\frac{1}{2}(3g_0-\ell_0) \le k \le -14\ell_0 + 15g_0 \tag{11}$$

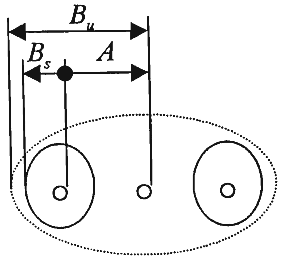


Fig.4 Schematic diagram of limit cycles

It is to be noted that the left-hand side condition is equal to the condition for every equilibrium point to be unstable. Now, since it is not easy to show the existence of an unstable limit surface, it will be shown that the trajectory starting from the inside of the unstable limit cycle does not diverge. Since it can be shown that there exists an unstable limit cycle, x_1 is bounded. Therefore, if $a_1 > 0$, all state variables are also bounded. This condition is satisfied since $G(s)$ has no pole in the RHP. The problem here is the accuracy of the describing function method. Although it is an approximate method, it has been known that if $G(s)$ has a sufficient low pass filter characteristic and nonlinear function has not a special form, then the results are reliable. It is to be noted that the condition (11) can also be applied to a nonlinear system with time lag.

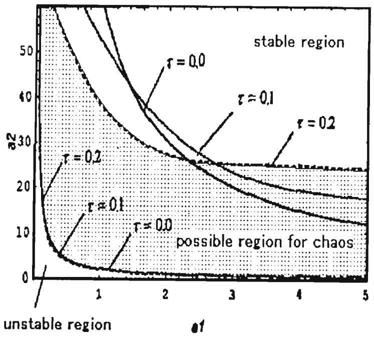
Example 3

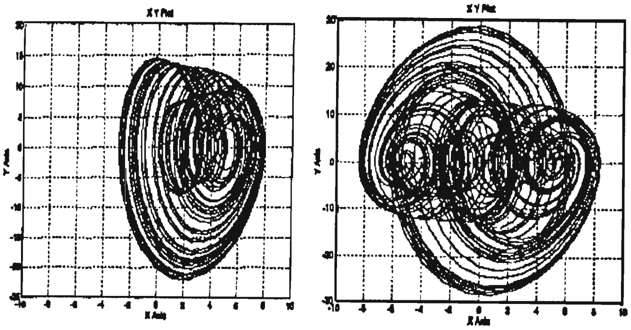
$$G(s) = \frac{\exp(-s\tau)}{s^3 + a_1 s^2 + a_2 s + a_3} \tag{12}$$

If $\tau = 0$, (11) becomes

$$\frac{a_1 a_2}{2} + a_3 \le k \le 14 a_1 a_2 + a_3 \tag{13}$$

The parameter regions for the occurrence of chaos for several values of τ will be shown in Fig.5. In the shaded region it is possible for chaos to occur, while stable limit cycles can also be observed. In the upper (lower) region of the shaded region, the state variables converge (diverge). By computer simulation the double and single scroll-type chaos can be observed as shown in Fig.6.





(a) $\tau=0.2, k=31, a_1=1.0, a_2=2.0, a_3=1.0$ (b) $\tau=0.2, k=31, a_1=1.0, a_2=1.9, a_3=1.0$

Fig.5 Parameter region for chaos for (12) (k=31, a_3 =1)

Fig.6 Chaotic trajectories

4. Conclusions

In this paper it was shown that the describing function method used in the control engineering is useful to analyze the bifurcation and chaotic phenomenon. The method is very simple and can be applied to a system of any order system and a system with time lag. The problem of using this method is the problem of accuracy, since the describing function method is an approximate method in itself. However, if the linear part of the system has a property of a low pass filter, it is known that the describing function method can be applied.

References

[1] A.Gelb and W.E.Vander Velde: Multiple-Input Describing Functions and Nonlinear System Desgin,(MacGraw-Hill, New York,1968).

[2] A.I.Mees: Dynamics of feedback systems, (John Wiley & Sons, 1981).

[3] R.Genesio and A.Tesi: Harmonic balance method for the analysis of chaotic dynamics in nonlinear systems, Automatica, vol.28, No.3, (1992), pp.531-548

[4] K.Hirai, M.Iwai and T.Ushio: Catastrophic jump phenomenon in a nonlinear control system, IEEE Trans. on Automatic Control, vol.AC-26, (1981), 601-603

[5] K.Hirai and N.Sawai: A general criterion for jump resonance of nonlinear control system, IEEE Trans. Automatic Control, AC-23,(1978), 896-901

[6] K.Hirai and N.Sawai: Island type jump resonance in nonlinear systems, Proc. Int. Conf. on Nonlinear Oscillation, Prague, (1978), 325-330

[7] K.Hirai: A simple criterion for the occurrence of chaos, Proc. of the Int. Conf. Nonlinearity, Bifurcation, and Chaos, Ludz, (1996), 133-136